Applied Mathematics and Computation 119 (2001) 265–281
www.elsevier.com/locate/amc

On stability of nonlinear differential systems via cone-valued Liapunov function method

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Abstract

The notions of partial, relative and ϕ_0-stability of ordinary differential equations (ODEs) are introduced in [E.P. Akpan, O. Akinyele, J. Math. Anal. Appl. 164 (1992) 307–324; V. Lakshmikantham, S. Leela, Differential and Integral Inequalities, vol. 1, Academic Press, New York, 1969]. In this paper, we extend these notions to new types of stability namely, ϕ_0-relative and ϕ_0-partial stability of ODEs using cone-valued Liapunov function method and comparison technique. © 2001 Published by Elsevier Science Inc.

1. Introduction

Lakshmikantham and Leela [3] initiated the development of a theory of differential inequalities through cone and cone-valued Liapunov function methods which depend on Liapunov's direct method (see [4]). They discussed and improved the notions of partial and relative stability of two differential systems.

The aim of this paper is to extend these notions to new types of stability, namely ϕ_0-relative stability of two differential systems:

$$\begin{aligned} x' &= f_1(t,x), \quad x(t_0) = x_0, \\ y' &= f_2(t,y), \quad y(t_0) = x_0, \end{aligned} \tag{1.1}$$

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PII: S0096-3003(99)00234-9

and ϕ_0-partial stability of two differential systems:

$$\begin{aligned} x' &= F(t,x,y), \quad x(t_0) = x_0, \\ y' &= H(t,x,y), \quad y(t_0) = y_0, \end{aligned} \tag{1.2}$$

where $f_1, f_2 \in C[\Re^+ \times \Re^n, \ \Re^n]$, and $f_1(t,0) = f_2(t,0) = 0$, $F \in C[\Re^+ \times \Re^n \times \Re^m, \ \Re^n]$, $H \in C[\Re^+ \times \Re^n \times \Re^m, \ \Re^m]$ and $F(t,0,0) = H(t,0,0) = 0$, $t \in \Re^+$, with $\Re^+ = [0,\infty)$, $\Re^n$ and $\Re^m$ are n- and m-dimensional real Euclidean spaces, respectively, with any convenient norm $\|\cdot\|$ and scalar product (,).

Recently, these systems were studied from other points of view by Lakshmikantham and Leela [3]. Furthermore these notions lie somewhere between the notion of ϕ_0-stability of Akpan and Akinyele [1] on one side and partial and relative stability of Lakshmikantham and Leela [3] on the other. The motivation of this work is the recent work of Akpan and Akinyele [1] and Akpan [2].

The following definitions will be needed in the sequel:

Definition 1.1 (Akpan and Akinyele [1]). A proper subset K_1 of $\Re^n$ is called a cone if

(i) $\lambda K_1 \subset K_1$, $\lambda \geqslant 0$,
(ii) $K_1 + K_1 \subset K$,
(iii) $\overline{K_1} = K_1$,
(iv) $K_1^\circ \neq \emptyset$,
(v) $K_1 \cap (-K_1) = \{0\}$,

where $\overline{K_1}$ and K_1° denote the closure and interior of K_1, respectively, and ∂K_1 denotes the boundary of K_1.

Definition 1.2 (Akpan and Akinyele [1]). The set $K_1^* = \{\phi \in \Re^n;\ (\phi,x) \geqslant 0,\ x \in K_1\}$ is called the adjoint cone if it satisfies properties (i)–(v) of Definition 1.1.

$$x \in \partial K_1 \quad \text{iff} \quad (\phi,x) = 0 \quad \text{for some } \phi \in K_{10}^*, \ \ K_{10} = K_1 \setminus \{0\}.$$

Definition 1.3 (Akpan and Akinyele [1]). A function $g : D \to \Re^n$, $D \subset \Re^n$ is called quasi-monotone relative to the cone K_1 if $x, y \in D$ and $y - x \in \partial K_1$, then there exists $\phi_0 \in K_{10}^*$ such that $(\phi_0, y - x) = 0$ and $(\phi_0, g(y) - g(x)) \geqslant 0$.

Definition 1.4 (Lakshmikantham and Leela [3]). A function $b(r)$ is said to belong to the class $\mathscr{K}$ if $b \in C[[0,\rho), \Re^+]$, $b(0) = 0$, and $b(r)$ is strictly monotone increasing function in r.

2. ϕ_0-Relative stability

In this section, we improve and extend the notion of ϕ_0-stability of the system (1.1) of Akpan and Akinyele [1] to a new type of stability, namely ϕ_0-

relative stability. The motivation of this work is the recent work of Akpan and Akinyele [1].

Following Lakshmikantham and Leela [3], for a Liapunov function $V(t,x,y) \in C[\Re^+ \times S_\rho \times S_\rho, K_1]$, for any $(t,x,y) \in \Re^+ \times S_\rho \times S_\rho$, $S_\rho = \{u \in \Re^n: \|u\| < \rho,\ \rho > 0\}$, $K_1 \subset \Re^n$ be a cone in $\Re^n$, and $V(t,x,y)$ has continuous partial derivatives with respect to t, x and y. We define $D^+V(t,x,y)$ by

$$D^+V(t,x,y) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} \cdot f_1(t,x) + \frac{\partial V}{\partial y} \cdot f_2(t,y),$$

where '$\cdot$' denotes the scalar product.

If $V(t,x,y)$ is locally Lipschitzian in x and y then the function $D^+V(t,x,y)$ is defined by

$$D^+V(t,x,y) = \lim_{h\to 0}\ \sup\ \frac{1}{h}[V(t+h, x+hf_1(t,x), y+hf_2(t,y)) - V(t,x,y)]$$

for $h > 0$.

Definition 2.1 (Lakshmikantham and Leela [3]). The two differential systems (1.1) are said to be relatively equistable, if for each, $\epsilon > 0$ and $t_0 \in \Re^+$, there exists a $\delta = \delta(t_0, \epsilon)$ which is continuous in t_0 for each ϵ such that the inequality

$$\|x_0 - y_0\| < \delta \quad \text{implies} \quad \|x(t) - y(t)\| < \epsilon, \qquad t \geqslant t_0$$

for any solutions $x(t) = x(t,t_0,x_0),\ y(t) = y(t,t_0,y_0)$ of (1.1).

Other relative stability concepts can be similarly defined (see [3]).

The following definitions are somewhat new and related with that of [1,3].

Definition 2.2. The two differential systems (1.1) are said to be relatively ϕ_0-equistable, if, for each $\epsilon > 0$, there exists $\delta = \delta(t_0,\epsilon)$, continuous in t_0 for each ϵ, such that for $\phi_0 \in K_0^*$ the inequality

$$(\phi_0, x_0 - y_0) < \delta \quad \text{implies} \quad (\phi_0, x^*(t) - y^*(t)) < \epsilon, \qquad t \geqslant t_0,$$

where $x^*(t)$ and $y^*(t)$ denote the maximal solutions of the two differential systems (1.1).

Definition 2.3. The two differential systems (1.1) are said to be uniformly relatively ϕ_0-stable if δ in Definition 2.2 is independent of t_0.

Definition 2.4. The two differential systems (1.1) are said to be relatively ϕ_0-equi-asymptotically stable if they are relatively ϕ_0-equistable, and for each $\epsilon > 0,\ t_0 \in \Re^+$, there exist positive numbers $\delta_0 = \delta_0(t_0)$ and $T = T(t_0,\epsilon)$ such that

$$(\phi_0, x_0 - y_0) < \delta_0 \quad \text{implies} \quad (\phi_0, x^*(t) - y^*(t)) < \epsilon, \qquad t \geqslant t_0 + T.$$

Definition 2.5. The two differential systems (1.1) are said to be uniformly relatively ϕ_0-asymptotically stable if δ_0 and T in Definition 2.4 are independent of t_0.

Consider the comparison differential system

$$u' = G(t, u), \quad u(t_0) = u_0 \geqslant 0, \ \ t_0 \geqslant 0, \tag{2.1}$$

where $G \in C[\mathfrak{R}^+ \times K_1, \mathfrak{R}^n]$ and K_1 is a cone in $\mathfrak{R}^n$. Let $r(t)$ be the maximal solution of (2.1).

Theorem 2.1. *Assume that there exist functions* $V(t,x,y) \in C[\mathfrak{R}^+ \times S_\rho \times S_\rho, K_1]$ *and* $f_1(t,x)$, $f_2(t,y) \in C[\mathfrak{R}^+ \times K_1, \mathfrak{R}^n]$ *such that* $V(t,x,x) = 0$, *and* $f_1(t,0) = f_2(t,0) = 0$, *satisfying*

(A_1) $V(t,x,y)$ *is locally Lipschitzian* x *and* y.

(A_2) $D^+V(t,x,y) \leqslant_{K_1} 0$.

(A_3) $f_1(t,x)$ *and* $f_2(t,y)$ *are quasi-monotone in* x, y, *respectively relative to* K_1.

(A_4) *For some* $\phi_0 \in K_{10}^*$ *and* $(t,x,y) \in \mathfrak{R}^+ \times S_\rho \times S_\rho$,

$$a[(\phi_0, x(t) - y(t))] \leqslant (\phi_0, V(t,x,y)), \quad a \in \mathscr{K}.$$

Then the two systems (1.1) *are relatively* ϕ_0-*equistable.*

Proof. Since $V(t,x,y)$ is continuous and $V(t,x,x) = 0$, for given $a_1(\epsilon) > 0$, $t_0 \in \mathfrak{R}^+$, there exists $\delta_1 = \delta_1(t_0, \epsilon) > 0$ such that

$$\|x_0 - y_0\| < \delta_1 \quad \text{implies} \quad \|V(t_0, x_0, y_0)\| \leqslant a_1(\epsilon), \qquad a_1 \in \mathscr{K}. \tag{2.2}$$

Now, for some $\phi_0 \in K_{10}^*$

$$\|\phi_0\| \|x_0 - y_0\| < \|\phi_0\| \delta_1 = \delta$$

implies

$$\|\phi_0\| \|V(t_0, x_0, y_0)\| < \|\phi_0\| a_1(\epsilon) = a(\epsilon).$$

Thus

$$(\phi_0, x_0 = y_0) \leqslant \delta \quad \text{implies} \quad (\phi_0, V(t_0, x_0, y_0)) \leqslant a(\epsilon), \qquad a \in \mathscr{K}, \tag{2.3}$$

where $\|\phi_0\| \delta_1 = \delta$ and $\|\phi_0\| a_1(\epsilon) = a(\epsilon)$. From ($A_2$), $V(t,x,y)$ is non-increasing, then

$$V(t,x,y) \leqslant V(t_0, x_0, y_0), \quad t \geqslant t_0. \tag{2.4}$$

From (2.3), (2.4), and condition (A_4), we get

$$(\phi_0, x_0 - y_0) < \delta$$

implies

$$a[(\phi_0, x(t) - y(t))] \leqslant (\phi_0, V(t,x,y)) \leqslant (\phi_0, V(t_0,x_0,y_0)) < a(\epsilon), \quad a \in \mathscr{K},$$

i.e.,

$$(\phi_0, x_0 - y_0) < \delta \quad \text{implies} \quad (\phi, x^*(t) - y^*(t)) < \epsilon, \qquad t \geqslant t_0.$$

Then the two systems (1.1) are relatively ϕ_0-equistable. □

Theorem 2.2. *Let the hypotheses of Theorem* 2.1 *be satisfied*, *except the condition* (A_4) *being replaced by*

(A_5) *For some* $\phi_0 \in K_{10}^*$ *and* $(t,x,y) \in \Re^+ \times S_\rho \times S_\rho$,

$$a[(\phi_0, x(t) - y(t))] \leqslant (\phi_0, V(t,x,y)) \leqslant b[(\phi_0, x(t) - y(t))], \quad a, b \in \mathscr{K}.$$

Then the two systems (1.1) *are uniformly relatively* ϕ_0*-stable.*

Proof. For $\epsilon > 0$, let $\delta = b^{-1}[a(\epsilon)]$ independent of t_0 for $a, b \in \mathscr{K}$ such that $(\phi_0, x_0 - y_0) < \delta$. Since by ($A_2$), $V(t,x,y)$ is non-increasing, it follows that

$$(\phi_0, V(t,x,y)) \leqslant (\phi_0, V(t_0,x_0,y_0)), \quad t \geqslant t_0. \tag{2.5}$$

From (2.5) and condition (A_5), we get

$$\begin{aligned} a[(\phi_0, x(t) - y(t))] &\leqslant (\phi_0, V(t,x,y)) \\ &\leqslant (\phi_0, V(t_0,x_0,y_0)) \leqslant b[(\phi_0, x(t) - y(t))] \\ &< b(\delta) = b[b^{-1}[a(\epsilon)]] = a(\epsilon), \end{aligned}$$

i.e., $(\phi_0, x(t) - y(t)) < \epsilon$.

Thus

$$(\phi_0, x_0 - y_0) < \delta \quad \text{implies} \quad (\phi_0, x^*(t) - y^*(t)) < \epsilon.$$

Then the two systems (1.1) are uniformly relatively ϕ_0-stable. □

Theorem 2.3. *Let the hypotheses of Theorem* 2.1 *be satisfied*, *except condition* (A_2) *being replaced by*

$$(A_6) \quad D^+(\phi_0, V(t,x,y)) \leqslant -c(\phi_0, V(t,x,y)), \quad c \in \mathscr{K}.$$

Then the two systems (1.1) *are relatively equi-asymptotically* ϕ_0*-stable.*

Proof. From Theorem 2.1, the two systems (1.1) are relatively ϕ_0-equistable. By assumption (A_6), $V(t,x,y)$ is monotone decreasing function, thus the limit

$$V^* = \lim_{t\to\infty} V(t,x,y)$$

exists. Now, we prove that $V^* = 0$. Suppose this is false, i.e. $V^* \neq 0$, then $c(V^*) \neq 0,\ c \in \mathscr{K}$. Since $c(r)$ is monotone increasing function, then

$$c[(\phi_0, V(t,x,y))] > c[(\phi_0, V^*)],$$

and so from (A_6), we get

$$D^+(\phi_0, V(t,x,y)) \leqslant -c[(\phi_0, V^*)], \quad c \in \mathscr{K}. \tag{2.6}$$

By integrating (2.6) on $[t_0, t]$, we obtain

$$(\phi_0, V(t,x,y)) \leqslant -c[(\phi_0, V^*)](t - t_0) + (\phi_0, V(t_0, x_0, y_0)).$$

Thus, as $t \to \infty$ and for some $\phi_0 \in K_0^*$, we get $(\phi_0, V(t,x,y)) \to -\infty$. This contradicts condition (A_4). Therefore, V^* must be equal to zero. Hence

$$(\phi_0, V(t,x,y)) \to 0 \quad \text{as } t \to \infty. \tag{2.7}$$

From (2.7) and condition (A_4), we get

$$(\phi_0, x(t) - y(t)) \to 0 \quad \text{as } t \to \infty.$$

Thus for given $\epsilon > 0$, $t_0 \in \mathfrak{R}^+$, there exists $\delta = \delta(t_0)$ and $T = T(t_0, \epsilon)$ such that

$$(\phi_0, x_0 - y_0) < \delta \quad \text{implies} \quad (\phi_0, x^*(t) - y^*(t)) < \epsilon, \qquad t \geqslant t_0 + T.$$

Then the two systems (1.1) are relatively ϕ_0-equi-asymptotically stable.

Theorem 2.4. *Let the hypotheses of Theorem* 2.2 *be satisfied, and condition* (A_2) *be replaced by*

$$(A_7) \quad D^+(\phi_0, V(t,x,y)) \leqslant -c[(\phi_0, x(t) - y(t))], \quad c \in \mathscr{K}.$$

Then the two systems (1.1) *are relatively uniformly ϕ_0-asymptotically stable.*

Proof. For given $\epsilon > 0$, choose $\delta = \delta(\epsilon)$ independent of t_0. Suppose that $(\phi_0, x_0 - y_0) < \delta$, then by Theorem 2.2 the two systems (1.1) are relatively ϕ_0-uniformly stable. Going through as in [1], we choose

$$V^* = \{\sup(\phi_0, V(t_0, x_0, y_0)) \colon (\phi_0, x_0 - y_0) < \delta\}$$

and

$$T(\epsilon) = V^*/c(\epsilon), \quad c \in \mathscr{K}.$$

Now, we prove that

$$(\phi_0, x_0 - y_0) < \delta \quad \text{implies} \quad (\phi_0, x(t) - y(t)) < \epsilon, \qquad t \geqslant t_0 + T. \tag{2.8}$$

Suppose that this is not true, then there exists at least one $t \geqslant t_0 + T(\epsilon)$ such that

$$(\phi_0, x_0 - y_0) < \delta \quad \text{implies} \quad (\phi_0, x(t) - y(t)) \geqslant \epsilon. \tag{2.9}$$

From (2.9), condition (A_7), and the monotoncity of c we get

$$D^+(\phi_0, V(t,x,y)) \leqslant -c(\epsilon). \tag{2.10}$$

By integrating (2.10) on $[t_0, t]$ we get

$$(\phi_0, V(t,x,y)) \leqslant (\phi_0, V(t_0,x_0,y_0)) - c(\epsilon)(t-t_0), \quad t \geqslant t_0 + T(\epsilon).$$

Thus for some $\phi_0 \in K_0^*$, as $t \to \infty$, we get

$$(\phi_0, V(t,x,y)) \to -\infty,$$

which contradicts condition (A_5). Hence for each $\epsilon > 0$, $t_0 \in \Re^+$, there exist positive numbers $\delta = \delta(\epsilon)$ and $T = T(\epsilon)$ such that

$$(\phi_0, x_0 - y_0) < \delta \quad \text{implies} \quad (\phi_0, x^*(t) - y^*(t)) < \epsilon, \qquad t \geqslant t_0 + T.$$

Then the two systems (1.1) are uniformly relatively ϕ_0-asymptotically stable. □

Theorem 2.5. *Assume that conditions* (A_1), (A_3) *and* (A_4) *of Theorem* 2.1 *are satisfied. Further assume that*

$$(A_8) \quad D^+V(t,x,y) \leqslant G(t, V(t,x,y)).$$

Then

(1) *If the zero solution of* (2.1) *is equistable, then the two systems* (1.1) *are relatively ϕ_0-equistable.*

(2) *If the zero solution of* (2.1) *is equi-asymptotically stable, then the two systems* (1.1) *are relatively equi-asymptotically ϕ_0-stable.*

Proof. (1) Since the zero solution of (2.1) is equistable, for $t_0 \in \Re^+$ and for given $a_1(\epsilon) > 0$, there exists a $\delta^* = \delta^*(t_0, \epsilon)$ such that

$$u_0 \leqslant \delta^* \quad \text{implies} \quad u(t, t_0, u_0) < a_1(\epsilon), \qquad a_1 \in \mathscr{K}. \tag{2.11}$$

Since $V(t,x,x) = 0$ and $V(t,x,y)$ is continuous, it follows that we find $\delta_1 = \delta_1(t_0, \epsilon)$ satisfying

$$\|x_0 - y_0\| \leqslant \delta_1, \quad V(t_0, x_0, y_0) < \delta^*.$$

Now, if we choose $V(t_0, x_0, y_0) = u_0$ and use assumption (A_8), we can apply Theorem 3.1 of [4] to obtain

$$V(t, x(t), y(t)) \leqslant r(t, t_0, u_0), \quad t \geqslant t_0, \tag{2.12}$$

where $r(t, t_0, u_0)$ is the maximal solution of (2.1), thus for some $\phi_0 \in K_0^*$,

$$\|\phi_0\| \|x_0 - y_0\| < \|\phi_0\|\delta_1 \quad \text{implies} \quad \|\phi_0\| \|V(t,x,y)\| < \|\phi_0\| a_1(\epsilon),$$

i.e.,

$$(\phi_0, x_0 - y_0) \leqslant \|\phi_0\| \|x_0 - y_0\| \leqslant \|\phi_0\|\delta_1 = \delta$$

implies

$$(\phi_0, V(t,x,y)) \leqslant \|\phi_0\| \|V(t,x,y)\| < \|\phi_0\| a_1(\epsilon) = a(\epsilon),$$

where $\|\phi_0\|\delta_1 = \delta$ and $\|\phi_0\|a_1(\epsilon) = a(\epsilon)$. It follows that

$$(\phi_0, x_0 - y_0) \leqslant \delta \quad \text{implies} \quad (\phi_0, V(t,x,y)) < a(\epsilon). \tag{2.13}$$

From (A_4) and by using (2.13), we get

$$a[(\phi_0, x(t) - y(t))] \leqslant (\phi_0, V(t,x,y)) < a(\epsilon), \quad t \geqslant t_0.$$

Hence

$$(\phi_0, x_0 - y_0) < \delta \quad \text{implies} \quad (\phi_0, x^*(t) - y^*(t)) < \epsilon, \qquad t \geqslant t_0,$$

where $x^*(t)$, $y^*(t)$ is the maximal solution of (1.1). Therefore the two differential systems (1.1) are relatively ϕ_0-equistable.

(2) Since the zero solution of (1.1) is equi-asymptotically stable, for given $a_1(\epsilon) > 0$, $t_0 \in \Re^+$, there exist positive numbers $\delta_0 = \delta_0(t_0)$ and $T = T(t_0, \epsilon)$ such that

$$u_0 \leqslant \delta_0 \quad \text{implies} \quad u(t, t_0, u_0) < a_1(\epsilon), \qquad t \geqslant t_0 + T.$$

We choose $\hat{\delta}_0 = \hat{\delta}_0(t_0)$. From the continuity of $V(t,x,y)$, we have

$$\|x_0 - y_0\| \leqslant \hat{\delta}_0 \quad \text{and} \quad V(t_0, x_0, y_0) \leqslant a_1(\epsilon),$$

as in (1), we get

$$V(t, x(t), y(t)) \leqslant r(t, t_0, u_0), \quad t \geqslant t_0.$$

Consequently, it follows that for some $\phi_0 \in K_0^*$,

$$(\phi_0, x_0 - y_0) \leqslant \|\phi_0\| \|x_0 - y_0\| < \|\phi_0\| \hat{\delta}_0 = \delta$$

implies that

$$(\phi_0, V(t,x,y)) \leqslant \|\phi_0\| \|V(t,x,y)\| < \|\phi_0\| a_1(\epsilon) = a(\epsilon), \quad t \geqslant t_0 + T,$$

where $\delta = \|\phi_0\| \hat{\delta}_0$ and $a(\epsilon) = \|\phi_0\| a_1(\epsilon)$, i.e.,

$$(\phi_0, x_0 - y_0) < \delta \quad \text{implies} \quad (\phi_0, V(t,x,y)) < a(\epsilon), \qquad t \geqslant t_0 + T. \tag{2.14}$$

From (A_4) and by using (2.14), we obtain the following inequality:

$$a[(\phi_0, x(t) - y(t))] \leqslant (\phi_0, V(t,x,y)) < a(\epsilon).$$

Hence

$$(\phi_0, x_0 - y_0) < \delta \quad \text{implies} \quad (\phi_0, x^*(t) - y^*(t)) < \epsilon \qquad \text{for } t \geqslant t_0 + T,$$

where $x^*(t)$, $y^*(t)$ is the maximal solution of (1.1). Then the two differential systems (1.1) are relatively equi-asymptotically ϕ_0-stable. □

The following definitions will be needed in the sequel.

Definition 2.6 (Akpan and Akinyele [1]). The zero solution of system (2.1) is said to be exponentially asymptotically ϕ_0-stable if there exist $\sigma > 0,\ \alpha > 0$ both real numbers such that

$$(\phi_0, r(t)) \leqslant \sigma(\phi_0, u_0) \exp[-\alpha(t - t_0)], \quad t \geqslant t_0,$$

where $r(t)$ is the maximal solution of (2.1).

Definition 2.7 (Lakshmikantham and Leela [3]). The two differential systems (1.1) are said to be relatively exponentially asymptotically stable if there exist $M > 0,\ \beta > 0$ both real numbers such that

$$\|x(t) - y(t)\| \leqslant M\|x_0 - y_0\| \exp[-\beta(t - t_0)], \quad t \geqslant t_0$$

for any solution $x(t)$ and $y(t)$ of (1.1).

Theorem 2.6. *Assume that conditions* (A_1) *and* (A_8) *are satisfied. Further assume that*

(A_9) $G(t,0) = 0,\ G(t,u)$ *is quasi-monotone in u relative to* K_1.

(A_{10}) *For some* $\phi_0 \in K_0^*,\ (t,x) \in \mathfrak{R}^+ \times S_\rho \times S_\rho$

$$b(\|x - y\|) \leqslant (\phi_0, V(t,x,y)) \leqslant a(t, \|x - y\|), \quad a, b \in \mathscr{K}.$$

Then

(3) *If the zero solution of system* (2.1) *is* ϕ_0*-equistable*, *then the two systems* (1.1) *are relatively equistable*.

(4) *If the zero solution of system* (2.1) *is quasi-equi-asymptotically* ϕ_0*-stable*, *then the two systems* (1.1) *are relatively equi-asymptotically stable*.

Proof. (3) Let the zero solution of (2.1) be ϕ_0-equistable and $0 < \epsilon < \rho$, $t \in \mathfrak{R}^+$. Then for given $b(\epsilon) > 0,\ t_0 \in \mathfrak{R}^+$ there exists $\delta = \delta(t_0, \epsilon) > 0$ such that

$$(\phi_0, u_0) \leqslant \delta \quad \text{implies} \quad (\phi_0, r(t)) < b(\epsilon).$$

Going through as in [1], we choose $a(t_0, \|x_0 - y_0\|) = (\phi_0, u_0)$, then from condition (A_{10}), we get

$$(\phi_0, V(t_0, x_0, y_0)) \leqslant a(t_0, \|x_0 - y_0\|) = (\phi_0, u_0).$$

Thus

$$V(t_0, x_0, y_0) \leqslant u_0.$$

By using condition (A_9), and applying Theorem 3.1 of [4] we obtain

$$V(t, x(t), y(t) \leqslant r(t, t_0, u_0), \quad t \geqslant t_0. \tag{2.15}$$

Now, we choose $a(t_0, \delta_1) = \delta,\ \delta_1 > 0$. Thus $\|x_0 - y_0\| \leqslant \delta_1$ and $a(t_0, \|x_0 - y_0\|) < \delta$ hold at the same time. Therefore from condition (A_{10}) and (2.15) we get

$$b(\|x-y\|) \leqslant (\phi_0, V(t,x,y)) \leqslant (\phi_0, r(t,t_0,u_0)) < b(\epsilon). \tag{2.16}$$

Thus

$$\|x_0 - y_0\| < \delta \quad \text{implies} \quad \|x(t) - y(t)\| < \epsilon, \qquad t \geqslant t_0,$$

where $x(t)$, $y(t)$ is any solution of the (1.1). Then the two systems (1.1) are relatively equistable.

(4) Since the zero solution of system (2.1) is quasi-equi-asymptotically ϕ_0-stable, then there exists as in [3] a positive function $\delta = \delta(t_0, \epsilon)$ for all $t \geqslant t_0 + T(\epsilon)$, such that

$$\|x_0 - y_0\| \leqslant \delta \quad \text{implies} \quad \|x(t) - y(t)\| < \epsilon, \qquad t \geqslant t_0 + T(\epsilon).$$

Suppose that this is false, then, going through as in the proof of Theorem 3.6 of [1], there exists a divergent sequence $\{t_k\}$, $t_k \geqslant t_0 + T(\epsilon)$ such that

$$\|x_0 - y_0\| \leqslant \delta \quad \text{implies} \quad \|x(t) - y(t)\| = \epsilon. \tag{2.17}$$

Then, from (2.15), (2.17) and (A_{10}) for $t = t_k$, we get the following contradiction:

$$\begin{aligned} b(\epsilon) = b(\|x(t_k) - y(t_k)\|) &\leqslant (\phi_0, V(t_k, x(t_k), y(t_k))) \\ &\leqslant (\phi_0, r(t_k, t_0, u_0)) < b(\epsilon). \end{aligned}$$

Therefore such a divergent sequence $\{t_k\}$ does not exist. Hence

$$\|x_0 - y_0\| < \delta \quad \text{implies} \quad \|x(t) - y(t)\| < \epsilon, \qquad t \geqslant t - 0 + T(\epsilon).$$

Then systems (1.1) are relatively equi-asymptotically stable. □

Theorem 2.7. *Let conditions* (A_1), (A_8) *and* (A_9) *be satisfied. Moreover assume that*

$$(\mathrm{A}_{11}) \quad \textit{For } c > 0,\ d > 0 (\phi_0, u_0) \leqslant \|x_0 - y_0\|^d$$

and

$$c\|x - y\|^d \leqslant (\phi_0, V(t,x,y)).$$

If the zero solution of system (2.1) *is exponentially asymptotically* ϕ_0*-stable, then the two systems* (1.1) *are relatively exponentially asymptotically stable.*

Proof. Since the zero solution of (2.1) is exponentially asymptotically ϕ_0-stable, there exist $\sigma > 0$ and $\alpha > 0$ both are real numbers such that

$$(\phi_0, r(t)) \leqslant \sigma(\phi_0, u_0)\ \exp[-\alpha(t - t_0)], \quad t \geqslant t_0. \tag{2.18}$$

Following [1], we let $x(t, t_0, x_0)$, and $y(t, t_0, y_0)$ be a solution of (1.1) such that $V(t_0, x_0, y_0) \leqslant u_0$, then by Theorem 3.1 of [4], we get

$$V(t,x(t),y(t)) \leqslant r(t). \tag{2.19}$$

Thus by condition (A_{11}) and (2.19), we get

$$c\|x-y\|^d \leqslant (\phi_0, V(t,x,y)) \leqslant (\phi_0, r(t)). \tag{2.20}$$

Thus, from inequalities (2.18) and (2.20) we obtain

$$c\|x-y\|^d \leqslant \sigma(\phi_0, u_0)\ \exp[-\alpha(t-t_0)], \quad t \geqslant t_0. \tag{2.21}$$

From condition (A_{11}) and (2.21), we get

$$\|x-y\| \leqslant M\|x_0-y_0\|\ \exp[-\beta(t-t_0)], \quad t \geqslant t_0,$$

where $M = (\sigma/c)^{1/d}$ and $\beta = \alpha/d$. Then the two systems (1.1) are relatively exponentially asymptotically stable. □

3. ϕ_0-Partial stability

In this section we discuss and extend the notion of partial stability of the two differential systems (1.2) to a new notion of stability, called ϕ_0-partial stability notion.

Following [1], let $K_1 \subset \Re^n$ be a cone in $\Re^n$ and $K_2 \subset \Re^m$ be a cone in $\Re^m$ satisfy properties (i)–(v) of Definition 1.1, it follows that $K = K_1 \cup K_2 \subset \Re^n \cup \Re^m$ be a cone in $\Re^n \cup \Re^m$.

The set K^* is called the adjoint cone if

$$K^* = \{\phi \in \Re^n \cup \Re^m\colon (\phi, x+y) \geqslant 0 \text{ for } x \in K_1 \subset K,\ \ y \in K_2 \subset K\}$$

and satisfies properties (i)–(v) of Definition 1.1, where $(\phi, x+y) \leq s\|\phi\|(\|x\| + \|y\|)$. For $m > n$ and $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_m)$, thus

$$\begin{aligned} x+y &= (x_1, x_2, \ldots, x_n, 0, 0, \ldots, 0) + (y_1, y_2, \ldots, y_m) \\ &= (x_1+y_1, x_2+y_2, \ldots, x_n+y_n, y_{n+1}, \ldots, y_m). \end{aligned}$$

Let $\Re^n = \{x \in \Re^n\colon \|x\| \leqslant \rho\}$, $\Re^m_p = \{x \in \Re^n\colon \|x\| \leqslant \rho\}$ and $V(t,x,y) \in C[\Re^+ \times \Re^n_\rho \times \Re^m_\rho, K]$, for any $(t,x,y) \in \Re^+ \times \Re^n_\rho \times \Re^m_\rho$, $V(t,0,0) = 0$, and $V(t,x,y)$ is locally Lipschitzian in x and y, we define the upper right-hand derivative of $V(t,x,y)$ by

$$\begin{aligned} D^+V(t,x,y) = \lim_{h\to 0}\ \sup \frac{1}{h}[V(t+h, x+hF(t,x,y),& \\ y+hH(t,x,y)) - V(t,x,y)]& \end{aligned}$$

(see [3]).

The following definition is somewhat new and related with that of [1,3].

Definition 3.1. The zero solution $x = 0, y = 0$ of (3.1) is said to be partially ϕ_0-equistable with respect to components x if for every $\epsilon > 0$, $t_0 \in \Re^+$, there exists a positive function $\delta = \delta(t_0, \epsilon)$ continuous in t_0 for each ϵ such that

$$(\phi_0, x_0 + y_0) < \delta \quad \text{implies} \quad (\phi_0, x^*(t, t_0, x_0, y_0)) < \epsilon, \qquad t \geqslant t_0,$$

where $x^*(t, t_0, x_0, y_0) = \max_{t \geqslant t_0} x(t, t_0, x_0, y_0),\ \phi_0 \in K_0^*$.

Other partial ϕ_0-stability notions can be similarly defined.

Theorem 3.1. *Assume that there exist functions* $V(t,x,y) \in C[\Re^+ \times \Re^n_\rho \times \Re^m_\rho, K]$, $F \in C[\Re^+ \times K_1 \times K_2, \Re^n]$ *and* $H \in C[\Re^+ \times K_1 \times K_2, \Re^m]$, *such that* $V(t,0,0) = 0$ *and* $F(t,0,0) = H(t,0,0) = 0$, *satisfying*

(H_1) *For some* $\phi_0 \in K_0^*$ *and* $(t,x,y) \in \Re^+ \times \Re^n_\rho \times \Re^m_\rho$,

$$a[(\phi_0, x^*(t))] \leqslant (\phi_0, V(t,x,y)), \quad a \in \mathscr{K},$$

(H_2) $V(t,x,y)$ *is locally Lipschitzian in* x *and* y,

(H_3) $F(t,x,y)$ *is quasi-monotone in* x *relative to* K_1,

(H_4) $D^+V(t,x,y) \leqslant 0$.

Then the zero solution of (1.2) *is partially* ϕ_0*-equistable with respect to* x.

Proof. Let $\epsilon > 0$, be given, $t_0 \in \Re^+$. Since $V(t,0,0) = 0$, and $V(t,x,y)$ is continuous function in t_0, then for given $a_1(\epsilon) > 0$, $t_0 \in \Re^+$, there exists a $\delta_1 = \delta_1(t_0, \epsilon)$ that is continuous in t_0 for each ϵ such that

$$\|x_0\| + \|y_0\| < \delta_1 \quad \text{implies} \quad \|V(t_0, x_0, y_0)\| < a_1(\epsilon), \qquad a_1 \in \mathscr{K}. \tag{3.1}$$

Thus, for some $\phi_0 \in K_0^*,\ t \geqslant t_0$

$$(\phi_0, x_0 + y_0) \leqslant \|\phi_0\|(\|x_0\| + \|y_0\|) \leqslant \|\phi_0\|\delta_1 = \delta$$

implies

$$(\phi_0, V(t_0, x_0, y_0)) \leqslant \|\phi_0\|\|V(t_0, x_0, y_0)\| \leqslant \|\phi_0\|a_1(\epsilon) = a(\epsilon), \tag{3.2}$$

where $\delta = \|\phi_0\|\delta_1$ and $a(\epsilon) = \|\phi_0\|a_1(\epsilon)$. From condition ($H_4$), we get

$$V(t,x,y) \leqslant V(t_0, x_0, y_0). \tag{3.3}$$

From (3.2), (3.3) and condition (H_1), we obtain

$$a[(\phi_0, x^*(t))] \leqslant (\phi_0, V(t,x,y)) \leqslant (\phi_0, V(t_0, x_0, y_0)) \leqslant a(\epsilon),$$

whenever $(\phi_0, x_0 + y_0) \leqslant \delta$.

Thus

$$(\phi_0, x_0 + y_0) \leqslant \delta \quad \text{implies} \quad (\phi_0, x^*(t)) < \epsilon, \qquad t \geqslant t_0.$$

Then the zero solution of (1.2) is ϕ_0-partially equistable. □

Theorem 3.2. *Let the hypotheses of Theorem* 3.1 *be satisfied, except condition* (H_1) *being replaced by*

(H_5) *For some* $\phi_0 \in K_0^*$ *and* $(t,x,y) \in \mathfrak{R}^+ \times \mathfrak{R}^n_\rho \times \mathfrak{R}^m_\rho$,

$$a[(\phi_0, x^*(t))] \leqslant (\phi_0, V(t,x,y)) \leqslant b[(\phi_0, x^*(t) + y^*(t))], \quad a,b \in \mathscr{K},$$

where $y^*(t) = \max_t \ y(t,t_0,x_0,y_0)$.

Then the zero solution of (1.2) *is uniformly partially* ϕ_0*-stable.*

Proof. Let $\epsilon > 0$ and choose $\delta = b^{-1}(a(\epsilon))$ independent of t_0. Now for some $\phi_0 \in K_0^*, \ a,b \in \mathscr{K}$, let $(\phi_0, x_0 + y_0) < \delta$. From condition (H_5), and (3.3), we get

$$a[(\phi_0, x^*(t))] \leqslant (\phi_0, V(t,x,y)) \leqslant (\phi_0, V(t_0,x_0,y_0)) \leqslant b[(\phi_0, x_0 + y_0)] \leqslant b(\delta) = b[b^{-1}[a(\epsilon)]] = a(\epsilon),$$

i.e.,

$$(\phi_0, x_0 + y_0) < \delta \quad \text{implies} \quad (\phi_0, x^*(t)) < \epsilon.$$

Then the zero solution of (1.2) is uniformly partially ϕ_0-stable. □

Theorem 3.3. *Let the conditions of Theorem* 3.2 *be satisfied except condition* (H_4) *being replaced by*

$$(\mathrm{H}_6) \quad D^+(\phi_0, V(t,x,y)) \leqslant -c(\phi_0, V(t,x,y)), \quad c \in \mathscr{K}.$$

Then the zero solution of system (1.2) *is partially equi-asymptotically* ϕ_0*-stable.*

Proof. Since condition (H_6) tends to condition (H_4), by applying Theorem 3.1, it follows that the zero solution of the system (1.2) is partially ϕ_0-equistable. From (H_6), $V(t,x,y)$ is monotone decreasing and hence $V^* = \lim_{t\to\infty} V(t,x,y)$ exists. Also we show that $V^* = 0$, if this not true, i.e., $V^* \neq 0$, then $c(V^*) \neq 0$, $c \in \mathscr{K}$. Since $c(r)$ is monotone increasing

$$c(\phi_0, V(t,x,y)) > c(\phi_0, V^*), \quad c \in \mathscr{K}.$$

Thus

$$D^+(\phi_0, V(t,x,y)) < -c(\phi_0, V^*). \tag{3.4}$$

By integration (3.4) on $[t_0, t]$, we obtain

$$(\phi_0, V(t,x,y)) \leqslant -c(\phi_0, V^*)(t - t_0) + (\phi_0, V(t_0,x_0,y_0)).$$

Hence for some $\phi_0 \in K_0^*$, we get $(\phi_0, V(t,x,y)) \to -\infty$ as $t \to \infty$ which contradicts condition (H_1). Thus $V^* = 0$, and hence

$$(\phi_0, V(t,x,y)) \to 0 \quad \text{as } t \to \infty.$$

By using the condition (H_5)

$$(\phi_0, x^*(t)) \to 0 \quad \text{as } t \to \infty.$$

Thus for given $\epsilon > 0$, $t_0 \in \Re^+$, there exist $\delta = \delta(t_0, \epsilon)$ and $T = T(t_0, \epsilon) > 0$, such that

$$(\phi_0, x_0 + y_0) < \delta \quad \text{implies} \quad (\phi_0, x^*(t)) < \epsilon, \qquad t \geqslant t_0 + T.$$

Then the zero solution of the system (1.2) is ϕ_0-partially equi-asymptotically stable. □

Theorem 3.4. *Let the conditions of Theorem* 3.2 *be satisfied except condition* (H_4) *which is being replaced by*

$$(H_7) \quad D^+(\phi_0, V(t,x,y)) \leqslant -c(\phi_0, x^*(t)), \quad c \in \mathscr{K}.$$

Then, the zero solution of (1.2) *is uniformly partially asymptotically* ϕ_0*-stable.*

Proof. The proof is similar to that of Theorem 2.4; so it is omitted. □

Theorem 3.5. *Let the conditions of Theorem* 3.1 *be satisfied except the condition* (H_4) *which is being replaced by*

$$(H_8) \quad D^+V(t,x,y) \leqslant G(t, V(t,x,y)), \ (t,x,y) \in \Re^+ \times \Re^n_\rho \times \Re^m_\rho.$$

Thus

(i) *If the zero solution of* (2.1) *is equistable, then the zero solution of* (1.2) *is partially* ϕ_0*-equistable.*

(ii) *If the zero solution of* (2.1) *is equi-asymptotically stable, then the zero solution of* (1.2) *is partially equi-asymptotically* ϕ_0*-stable.*

Proof. (i) Since the zero solution of (2.1) is equistable, for $t_0 \in \Re^+$, and for given $a_1(\epsilon) > 0$ there exists a $\delta^* = \delta^*(t_0, \epsilon)$ such that

$$u_0 \leqslant \delta^* \quad \text{implies} \quad u(t, t_0, u_0) < a_1(\epsilon), \qquad a_1 \in \mathscr{K}. \tag{3.5}$$

Since $V(t,0,0) = 0$ and the function $V(t,x,y)$ is continuous, there exists a $\delta_1 = \delta_1(t_0, \epsilon)$, such that

$$\|x_0\| + \|y_0\| \leqslant \delta_1, \quad \|V(t_0, x_0, y_0)\| < \delta^*.$$

satisfying at the same time. Choose $u_0 = V(t_0, x_0, y_0)$ and condition (H_8). Therefore we can apply Theorem 3.1 of [4] to obtain

$$V(t, x(t), y(t)) \leqslant r(t, t_0, u_0), \quad t \geqslant t_0, \tag{3.6}$$

where $r(t, t_0, u_0)$ is the maximal solution of (2.1). Now for some $\phi_0 \in K_0^*$,

$$(\phi_0, x_0 + y_0) \leqslant \|\phi_0\|(\|x_0 + y_0\|) < \|\phi_0\|\delta_1 = \delta$$

implies

$$(\phi_0), V(t,x,y)) \leqslant \|\phi_0\| \|V(t,x,y)\| < \|\phi_0\| a_1(\epsilon) = a(\epsilon), \tag{3.7}$$

where $\|\phi_0\|\delta_1 = \delta$ and $\|\phi_0\|a_1(\epsilon) = a(\epsilon)$. From (3.7) and condition (H_1), we obtain

$$a[(\phi_0, x(t))] \leqslant (\phi_0, V(t,x,y)) < a(\epsilon), \quad t \geqslant t_0.$$

Whenever $(\phi_0, x_0 + y_0) < \delta$,

$$(\phi_0, x_0 + y_0) < \delta \quad \text{implies} \quad (\phi_0, x^*(t)) < \epsilon, \qquad t \geqslant t_0.$$

Then the zero solution of (1.2) is partially ϕ_0-equistable.

(ii) The proof of part (ii) is similar to the proof of part (2) of Theorem 2.4; so it is omitted. □

Theorem 3.6. *Let conditions* (A_8), (H_2) *and* (H_8) *be satisfied. Further assume that*

(H_9) *For some* $\phi_0 \in K_0^*, (t,x,y) \in \Re^+ \times \Re_\rho^n \times \Re_\rho^m$

$$a(\|x^*\|) \leqslant (\phi_0, V(t,x,y)) \leqslant b(t, \|x\| + \|y\|), \quad a, b \in \mathscr{K}.$$

Then

(iii) *If the zero solution of* (2.1) *is* ϕ_0-*equistable, then the zero solution of* (1.2) *is partially equistable with respect to x.*

(iv) *If the zero solution of* (2.1) *is quasi-equi-asymptotically* ϕ_0-*stable, then the zero solution of* (1.2) *is partially quasi-equi-asymptotically stable.*

Proof. (iii) Let the zero solution of (2.1) be ϕ_0-equistable, and $0 < \epsilon < \rho,\ t \in \Re^+$. Then for given $a(\epsilon) > 0,\ t_0 \in \Re^+$, there exists $\delta = \delta(t_0, \epsilon) > 0$, such that

$$(\phi_0, u_0) < \delta \quad \text{implies} \quad (\phi_0, r(t)) < b(\epsilon), \tag{3.8}$$

where $r(t)$ is the maximal solution of (2.1). Going through as in the proof of Theorem 3.6 of [1], we choose $b(t_0, \|x_0\| + \|y_0\|) = (\phi_0, u_0)$, then from the condition (H_9), we get

$$(\phi_0, V(t_0, x_0, y_0)) \leqslant b(t_0, \|x_0\| + \|y_0\|) = (\phi_0, u_0).$$

Thus

$$V(t_0, x_0, y_0) \leqslant u_0.$$

By using the condition (H_8) and applying Theorem 3.1 in [4] we obtain

$$V(t, x(t), y(t)) \leqslant r(t, t_0, u_0), \quad t \geqslant t_0. \tag{3.9}$$

Now, choose $\delta_1 > 0$ such that $b(t_0, \delta_1) = \delta$. Thus the inequalities $\|x_0\| + \|y_0\| \leqslant \delta_1$ and $b(t_0, \|x_0\| + \|y_0\|) < \delta$ hold together. Therefore from condition (H_9), (3.8), and (3.9) we get

$$a(\|x\|) \leqslant (\phi_0, V(t,x,y)) \leqslant (\phi_0, r(t,t_0,u_0)) < a(\epsilon). \tag{3.10}$$

Thus

$$\|x_0\| + \|y_0\| < \delta_1 \quad \text{implies} \quad \|x^*(t)\| < \epsilon, \qquad t \geqslant t_0.$$

Then the zero solution of (1.2) is partially equistable with respect to x.

The remainder of the proof is similar to part (4) of Theorem 2.6 and, therefore, omitted. □

Theorem 3.7. *Let conditions* (A_8), (H_2) *and* (H_8) *be satisfied. Furthermore assume that*

(H_{10}) *For* $c > 0$, $d > 0$, $(\phi_0, u_0) \leqslant (\|x_0\| + \|y_0\|)^d$

and

$$c\|x\|^d \leqslant (\phi_0, V(t,x,y)).$$

If the zero solution of system (2.1) *is exponentially asymptotically* ϕ_0*-stable, then the zero solution of* (1.2) *is partially exponentially asymptotically stable.*

Proof. Since the zero solution of (2.1) is exponentially asymptotically ϕ_0-stable, then there exist $\sigma > 0$ and $\alpha > 0$ both real numbers such that

$$(\phi_0, r(t)) \leqslant \sigma(\phi_0, u_0)\exp[-\alpha(t-t_0)], \quad t \geqslant t_0. \tag{3.11}$$

Following [1] we let $x(t,t_0,x_0)$, and $y(t,t_0,y_0)$ be any solution of (1.2), such that $V(t_0,x_0,y_0) \leqslant u_0$, then by Theorem 3.1 of [4], we get

$$V(t,x(t),y(t)) \leqslant r(t). \tag{3.12}$$

Thus by condition (H_{10}) and (3.12) we get

$$c\|x\|^d \leqslant (\phi_0, V(t,x,y)) \leqslant (\phi_0, r(t)). \tag{3.13}$$

Thus, from inequalities (3.11) and (3.13), we get

$$c\|x\|^d \leqslant (\phi_0, r(t)) \leqslant \sigma(\phi_0, u_0)\exp[-\alpha(t-t_0)], \quad t \geqslant t_0. \tag{3.14}$$

From condition (H_{10}) and (3.14), we get

$$\|x\| \leqslant M\|x_0\| + \|y_0\|\exp[-\beta(t-t_0)], \quad t \geqslant t_0,$$

where $M = (\sigma/c)^{1/d}$ and $\beta = \alpha/d$. Then the zero solution of (1.2) is partially exponentially asymptotically stable. □

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